Environments

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Term 3 2024

Where we're at

- We refined the abstract M-Machine to a C-Machine, with explicit stacks:

$$s \succ e \qquad s \prec v$$

- Function application is still executed via substitution:

$$\overline{(\texttt{Apply}\ \langle\!\langle f.x.\ e\rangle\!\rangle\ \square) \triangleright s \prec v\ \mapsto_C\ s \succ e[x := v, f := (\texttt{Fun}\ (f.x.e))]}$$

- We're going to extend our C-Machine to replace substitutions with an environment, giving us a new *E-Machine*

Environments

Definition

An *environment* is a context containing the values of variables.

It is like the states of TinyImp, except the value of a variable never changes.

$$\frac{}{\bullet\ \textbf{Env}} \qquad \frac{\eta\ \textbf{Env}}{x = v, \eta\ \textbf{Env}}$$

$\eta(x)$ denotes the leftmost value bound to to x in η.

Let's change our machine states to include an environment:

$$s \mid \eta \succ e \qquad s \mid \eta \prec v$$

First Attempt

First, we'll add a rule for consulting the environment if we encounter a free variable:

$$\frac{}{s \mid \eta \succ x \quad \mapsto_E \quad s \mid \eta \prec \eta(x)}$$

Then, we just need to handle function application.

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One broken attempt:

$$\frac{}{(\texttt{Apply}\ \langle\!\langle f.x.\ e \rangle\!\rangle\ \square) \triangleright s \mid \eta \prec v \ \mapsto_E \ s \mid (x = v, f = \langle\!\langle f.x.\ e \rangle\!\rangle, \eta) \succ e}$$

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We don't know when to remove the variables again!

Second Attempt

We will extend our stacks to allow us to save the old environment to it.

$$\frac{\eta\ \textbf{Env} \quad s\ \textbf{Stack}}{\eta \triangleright s\ \textbf{Stack}}$$

When we call a function, we save the environment to the stack.

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When the function returns, we restore the old environment, clearing out the new bindings:

$$\frac{}{\eta \triangleright s \mid \eta' \prec v \ \mapsto_E\ s \mid \eta \prec v}$$

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$$\frac{}{\eta \triangleright s \mid \eta' \prec v \ \mapsto_E\ s \mid \eta \prec v}$$

This attempt is also broken (we'll see why soon)

Simple Example

$$\circ \quad | \quad \bullet \qquad\qquad \succ \quad (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Plus}\ x\ (\mathtt{N}\ 1))))$$

Simple Example

$$\circ \mid \bullet \succ \texttt{(Ap (Fun (}f.x.\texttt{ (Plus }x\texttt{ (N 1))))}$$
$$\texttt{(Ap }\Box\texttt{ (N 3))} \triangleright \circ \mid \bullet \succ \texttt{(Fun (}f.x.\texttt{ (Plus }x\texttt{ (N 1))))}$$

Simple Example

$$
\begin{array}{rcl@{\quad}l}
\circ & \mid & \bullet & \succ \; (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Plus}\ x\ (\mathtt{N}\ 1))))) \\
(\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright \circ & \mid & \bullet & \succ \; (\mathtt{Fun}\ (f.x.\ (\mathtt{Plus}\ x\ (\mathtt{N}\ 1)))) \\
(\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright \circ & \mid & \bullet & \prec \; \langle\!\langle f.x.\ (\mathtt{Plus}\ x\ (\mathtt{N}\ 1)) \rangle\!\rangle
\end{array}
$$

Simple Example

$$
\begin{array}{rcl}
\circ & | & \bullet \\
(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright \circ & | & \bullet \\
(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright \circ & | & \bullet \\
(\texttt{Ap}\ \langle\!\langle \cdots \rangle\!\rangle\ \square) \triangleright \circ & | & \bullet
\end{array}
\quad
\begin{array}{ll}
\succ & (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1))))) \\
\succ & (\texttt{Fun}\ (f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1)))) \\
\prec & \langle\!\langle f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1)) \rangle\!\rangle \\
\succ & (\texttt{N}\ 3)
\end{array}
$$

Simple Example

$$
\begin{array}{rcl|cl}
\circ & \mid & \bullet & \succ & (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1))))) \\
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(\texttt{Ap}\ \langle\!\langle \cdots \rangle\!\rangle\ \square) \triangleright \circ & \mid & \bullet & \succ & (\texttt{N}\ 3) \\
(\texttt{Ap}\ \langle\!\langle \cdots \rangle\!\rangle\ \square) \triangleright \circ & \mid & \bullet & \prec & 3
\end{array}
$$

Simple Example

$$
\begin{array}{rcl@{\quad}c@{\quad}l}
\circ & | & \bullet & \succ & (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1))))) \\
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\bullet \triangleright \circ & | & x = 3, f = \langle\langle\cdots\rangle\rangle, \bullet & \succ & (\texttt{Plus}\ x\ (\texttt{N}\ 1))
\end{array}
$$

Simple Example

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\begin{array}{rcl@{\quad}c@{\quad}l}
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\bullet \triangleright \circ & \mid & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \succ & (\mathtt{Plus}\ x\ (\mathtt{N}\ 1)) \\
(\mathtt{Plus}\ \square\ (\mathtt{N}\ 1)) \triangleright \bullet \triangleright \circ & \mid & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \succ & x
\end{array}
$$

Simple Example

$$
\begin{array}{rclcl}
\circ & | & \bullet & \succ & (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1))))) \\
(\texttt{Ap}\ \Box\ (\texttt{N}\ 3)) \triangleright \circ & | & \bullet & \succ & (\texttt{Fun}\ (f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1)))) \\
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\bullet \rhd \circ & | & x = 3, f = \langle\!\langle\cdots\rangle\!\rangle, \bullet & & \succ & (\texttt{Plus}\ x\ (\texttt{N}\ 1)) \\
(\texttt{Plus}\ \Box\ (\texttt{N}\ 1)) \rhd \bullet \rhd \circ & | & x = 3, f = \langle\!\langle\cdots\rangle\!\rangle, \bullet & & \succ & x \\
(\texttt{Plus}\ \Box\ (\texttt{N}\ 1)) \rhd \bullet \rhd \circ & | & x = 3, f = \langle\!\langle\cdots\rangle\!\rangle, \bullet & & \prec & 3 \\
(\texttt{Plus}\ 3\ \Box) \rhd \bullet \rhd \circ & | & x = 3, f = \langle\!\langle\cdots\rangle\!\rangle, \bullet & & \succ & (\texttt{N}\ 1)
\end{array}
$$

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\bullet \rhd \circ & \mid x = 3, f = \langle\!\langle\cdots\rangle\!\rangle, \bullet & \succ & (\texttt{Plus}\ x\ (\texttt{N}\ 1)) \\
(\texttt{Plus}\ \square\ (\texttt{N}\ 1)) \rhd \bullet \rhd \circ & \mid x = 3, f = \langle\!\langle\cdots\rangle\!\rangle, \bullet & \succ & x \\
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(\texttt{Plus}\ 3\ \square) \rhd \bullet \rhd \circ & \mid x = 3, f = \langle\!\langle\cdots\rangle\!\rangle, \bullet & \prec & 1
\end{array}
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\bullet \rhd \circ & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \succ & (\mathtt{Plus}\ x\ (\mathtt{N}\ 1)) \\
(\mathtt{Plus}\ \square\ (\mathtt{N}\ 1)) \rhd \bullet \rhd \circ & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \succ & x \\
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(\mathtt{Plus}\ 3\ \square) \rhd \bullet \rhd \circ & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \succ & (\mathtt{N}\ 1) \\
(\mathtt{Plus}\ 3\ \square) \rhd \bullet \rhd \circ & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \prec & 1 \\
\bullet \rhd \circ & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \prec & 4
\end{array}
$$

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\begin{array}{rclcl}
\circ & | & \bullet & \succ & (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Plus}\ x\ (\mathtt{N}\ 1))))) \\
(\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright \circ & | & \bullet & \succ & (\mathtt{Fun}\ (f.x.\ (\mathtt{Plus}\ x\ (\mathtt{N}\ 1)))) \\
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(\mathtt{Plus}\ 3\ \square) \triangleright \bullet \triangleright \circ & | & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \succ & (\mathtt{N}\ 1) \\
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\circ & | & \bullet & \prec & 4
\end{array}
$$

Simple Example

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\begin{array}{rl|ll}
\circ & & \bullet & \succ \ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1))))) \\
(\texttt{Ap}\ \Box\ (\texttt{N}\ 3)) \triangleright \circ & & \bullet & \succ \ (\texttt{Fun}\ (f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1)))) \\
(\texttt{Ap}\ \Box\ (\texttt{N}\ 3)) \triangleright \circ & & \bullet & \prec \ \langle\!\langle f.x.\ (\texttt{Plus}\ x\ (\texttt{N}\ 1)) \rangle\!\rangle \\
(\texttt{Ap}\ \langle\!\langle \cdots \rangle\!\rangle\ \Box) \triangleright \circ & & \bullet & \succ \ (\texttt{N}\ 3) \\
(\texttt{Ap}\ \langle\!\langle \cdots \rangle\!\rangle\ \Box) \triangleright \circ & & \bullet & \prec \ 3 \\
\bullet \triangleright \circ & & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \succ \ (\texttt{Plus}\ x\ (\texttt{N}\ 1)) \\
(\texttt{Plus}\ \Box\ (\texttt{N}\ 1)) \triangleright \bullet \triangleright \circ & & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \succ \ x \\
(\texttt{Plus}\ \Box\ (\texttt{N}\ 1)) \triangleright \bullet \triangleright \circ & & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \prec \ 3 \\
(\texttt{Plus}\ 3\ \Box) \triangleright \bullet \triangleright \circ & & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \succ \ (\texttt{N}\ 1) \\
(\texttt{Plus}\ 3\ \Box) \triangleright \bullet \triangleright \circ & & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \prec \ 1 \\
\bullet \triangleright \circ & & x = 3, f = \langle\!\langle \cdots \rangle\!\rangle, \bullet & \prec \ 4 \\
\circ & & \bullet & \prec \ 4
\end{array}
$$

Seems to work for basic examples, but is there some way to break it?

Closure Capture

$$\circ \mid \bullet \succ \texttt{(Ap (Ap (Fun (}f.x.\ \texttt{(Fun (}g.y.\ x\texttt{)))) (N 3)) (N 4))}$$

Closure Capture

$$\circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4))$$

$$\mapsto_E \quad (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))$$

Closure Capture

$$
\begin{aligned}
& \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))\ (\mathtt{N}\ 4)) \\
\mapsto_E\ & (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3)) \\
\mapsto_E\ & (\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))
\end{aligned}
$$

Closure Capture

$$
\begin{array}{ll}
& \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))\ (\mathtt{N}\ 4)) \\
\mapsto_E & (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3)) \\
\mapsto_E & (\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x)))) \\
\mapsto_E & (\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle f.x.\ (\mathtt{Fun}\ (g.y.\ x)) \rangle\!\rangle
\end{array}
$$

Closure Capture

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\begin{array}{ll}
 & \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))\ (\mathtt{N}\ 4)) \\
\mapsto_E & (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3)) \\
\mapsto_E & (\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x)))) \\
\mapsto_E & (\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle f.x.\ (\mathtt{Fun}\ (g.y.\ x)) \rangle\!\rangle \\
\mapsto_E & (\mathtt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{N}\ 3)
\end{array}
$$

Closure Capture

$$
\begin{aligned}
& \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4)) \\
\mapsto_E \quad & (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3)) \\
\mapsto_E \quad & (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x)))) \\
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\mapsto_E \quad & (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 3) \\
\mapsto_E \quad & (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3
\end{aligned}
$$

Closure Capture

$$
\begin{array}{ll}
& \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4)) \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3)) \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x)))) \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle f.x.\ (\texttt{Fun}\ (g.y.\ x))\rangle\!\rangle \\
\mapsto_E & (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 3) \\
\mapsto_E & (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3 \\
\mapsto_E & \bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \succ (\texttt{Fun}\ (g.y.\ x))
\end{array}
$$

Closure Capture

$$\circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4))$$

$$\mapsto_E\quad (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))$$

$$\mapsto_E\quad (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))$$

$$\mapsto_E\quad (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle f.x.\ (\texttt{Fun}\ (g.y.\ x))\rangle\!\rangle$$

$$\mapsto_E\quad (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 3)$$

$$\mapsto_E\quad (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3$$

$$\mapsto_E\quad \bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \succ (\texttt{Fun}\ (g.y.\ x))$$

$$\mapsto_E\quad \bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \prec \langle\!\langle g.y.\ x\rangle\!\rangle$$

Closure Capture

$$\circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4))$$

$$\mapsto_E\ (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))$$

$$\mapsto_E\ (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))$$

$$\mapsto_E\ (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle f.x.\ (\texttt{Fun}\ (g.y.\ x))\rangle\!\rangle$$

$$\mapsto_E\ (\texttt{Ap}\ \langle\!\langle f \cdots\rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 3)$$

$$\mapsto_E\ (\texttt{Ap}\ \langle\!\langle f \cdots\rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3$$

$$\mapsto_E\ \bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots\rangle\!\rangle, \bullet \succ (\texttt{Fun}\ (g.y.\ x))$$

$$\mapsto_E\ \bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots\rangle\!\rangle, \bullet \prec \langle\!\langle g.y.\ x\rangle\!\rangle$$

$$\mapsto_E\ (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle g.y.\ x\rangle\!\rangle$$

Closure Capture

$$
\begin{array}{ll}
& \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4)) \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3)) \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x)))) \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle f.x.\ (\texttt{Fun}\ (g.y.\ x)) \rangle\!\rangle \\
\mapsto_E & (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 3) \\
\mapsto_E & (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3 \\
\mapsto_E & \bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \succ (\texttt{Fun}\ (g.y.\ x)) \\
\mapsto_E & \bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \prec \langle\!\langle g.y.\ x \rangle\!\rangle \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle g.y.\ x \rangle\!\rangle \\
\mapsto_E & (\texttt{Ap}\ \langle\!\langle g.y.\ x \rangle\!\rangle\ \square) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 4)
\end{array}
$$

Closure Capture

$$
\begin{array}{ll}
 & \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4)) \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3)) \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x)))) \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle f.x.\ (\texttt{Fun}\ (g.y.\ x)) \rangle\!\rangle \\
\mapsto_E & (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 3) \\
\mapsto_E & (\texttt{Ap}\ \langle\!\langle f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3 \\
\mapsto_E & \bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \succ (\texttt{Fun}\ (g.y.\ x)) \\
\mapsto_E & \bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \prec \langle\!\langle g.y.\ x \rangle\!\rangle \\
\mapsto_E & (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle g.y.\ x \rangle\!\rangle \\
\mapsto_E & (\texttt{Ap}\ \langle\!\langle g.y.\ x \rangle\!\rangle\ \square) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 4) \\
\mapsto_E & (\texttt{Ap}\ \langle\!\langle g.y.\ x \rangle\!\rangle\ \square) \triangleright \circ \mid \bullet \prec 4
\end{array}
$$

Closure Capture

$$
\begin{aligned}
& \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))\ (\mathtt{N}\ 4)) \\
\mapsto_E\ & (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3)) \\
\mapsto_E\ & (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x)))) \\
\mapsto_E\ & (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle f.x.\ (\mathtt{Fun}\ (g.y.\ x))\rangle\!\rangle \\
\mapsto_E\ & (\mathtt{Ap}\ \langle\!\langle f \cdots\rangle\!\rangle\ \Box) \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{N}\ 3) \\
\mapsto_E\ & (\mathtt{Ap}\ \langle\!\langle f \cdots\rangle\!\rangle\ \Box) \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3 \\
\mapsto_E\ & \bullet \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots\rangle\!\rangle, \bullet \succ (\mathtt{Fun}\ (g.y.\ x)) \\
\mapsto_E\ & \bullet \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots\rangle\!\rangle, \bullet \prec \langle\!\langle g.y.\ x\rangle\!\rangle \\
\mapsto_E\ & (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle g.y.\ x\rangle\!\rangle \\
\mapsto_E\ & (\mathtt{Ap}\ \langle\!\langle g.y.\ x\rangle\!\rangle\ \Box) \triangleright \circ \mid \bullet \succ (\mathtt{N}\ 4) \\
\mapsto_E\ & (\mathtt{Ap}\ \langle\!\langle g.y.\ x\rangle\!\rangle\ \Box) \triangleright \circ \mid \bullet \prec 4 \\
\mapsto_E\ & \bullet \triangleright \circ \mid y = 4, g = \langle\!\langle g.y.\ x\rangle\!\rangle, \bullet \succ x
\end{aligned}
$$

Oh no! We're stuck!

Something went wrong!

When we return functions, the function's body escapes the scope of bound variables from where it as defined:

$$(\textbf{let } x = 3 \textbf{ in recfun } f\ y = x + y)\ 5$$

The function value $\langle\langle f.y.\ x + y\rangle\rangle$, when it is applied, does not "remember" that $x = 3$.

Something went wrong!

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$$(\textbf{let } x = 3 \textbf{ in recfun } f \ y = x + y)\ 5$$

The function value $\langle\!\langle f.y.\ x + y \rangle\!\rangle$, when it is applied, does not "remember" that $x = 3$.

Solution: Store the environment inside the function value!

$$\frac{}{s \mid \eta \succ (\texttt{Recfun}\ (f.x.\ e)) \ \mapsto_E s \mid \eta \prec \langle\!\langle \eta,\ f.x.\ e \rangle\!\rangle}$$

This type of function value is called a *closure*.

$$\frac{}{(\texttt{Apply}\ \langle\!\langle \eta', f.x.\ e\rangle\!\rangle\ \square) \triangleright s \mid \eta \prec v \ \mapsto_E\ \eta \triangleright s \mid (x = v, f = \langle\!\langle f.x.\ e\rangle\!\rangle, \eta') \succ e}$$

$$\frac{}{(\texttt{Apply}\ \langle\!\langle \eta', f.x.\ e \rangle\!\rangle\ \square) \triangleright s \mid \eta \prec v \ \mapsto_E\ \eta \triangleright s \mid (x = v, f = \langle\!\langle f.x.\ e \rangle\!\rangle, \eta') \succ e}$$

Store the old env. in the stack

$$\frac{}{(\mathtt{Apply}\ \langle\!\langle \eta', f.x.\ e \rangle\!\rangle\ \square) \triangleright s \mid \eta \prec v \ \mapsto_E\ \eta \triangleright s \mid (x = v, f = \langle\!\langle f.x.\ e \rangle\!\rangle, \eta') \succ e}$$

Store the old env. in the stack

Retrieve the new env. from the closure

$\circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4))$

$(\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))$

$(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))$

$$\circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4))$$

$$(\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))$$

$$(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))$$

$$(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle \bullet, f.x.\ (\texttt{Fun}\ (g.y.\ x)) \rangle\!\rangle$$

$$(\texttt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 3)$$

$$(\texttt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3$$

$\circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x)))) (\texttt{N}\ 3)) (\texttt{N}\ 4))$

$(\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x)))) (\texttt{N}\ 3))$

$(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))$

$(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle \bullet, f.x.\ (\texttt{Fun}\ (g.y.\ x)) \rangle\!\rangle$

$(\texttt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 3)$

$(\texttt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3$

$\bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \succ (\texttt{Fun}\ (g.y.\ x))$

$$\circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))\ (\mathtt{N}\ 4))$$

$$(\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))$$

$$(\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))$$

$$(\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle \bullet, f.x.\ (\mathtt{Fun}\ (g.y.\ x)) \rangle\!\rangle$$

$$(\mathtt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \square) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{N}\ 3)$$

$$(\mathtt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \square) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3$$

$$\bullet \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \succ (\mathtt{Fun}\ (g.y.\ x))$$

$$\bullet \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \prec \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x \rangle\!\rangle$$

$$\circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))\ (\mathtt{N}\ 4))$$
$$(\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))$$
$$(\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))$$
$$(\mathtt{Ap}\ \square\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle \bullet, f.x.\ (\mathtt{Fun}\ (g.y.\ x))\rangle\!\rangle$$
$$(\mathtt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \square) \triangleright (\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{N}\ 3)$$
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$$(\mathtt{Ap}\ \square\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x\rangle\!\rangle$$
$$(\mathtt{Ap}\ \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x\rangle\!\rangle\ \square) \triangleright \circ \mid \bullet \succ (\mathtt{N}\ 4)$$
$$(\mathtt{Ap}\ \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x\rangle\!\rangle\ \square) \triangleright \circ \mid \bullet \prec 4$$

$$\circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))\ (\mathtt{N}\ 4))$$
$$(\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Ap}\ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))\ (\mathtt{N}\ 3))$$
$$(\mathtt{Ap}\ \Box\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{Fun}\ (f.x.\ (\mathtt{Fun}\ (g.y.\ x))))$$
$$(\mathtt{Ap}\ \Box\ (\mathtt{N}\ 3)) \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle \bullet, f.x.\ (\mathtt{Fun}\ (g.y.\ x)) \rangle\!\rangle$$
$$(\mathtt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \Box) \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\mathtt{N}\ 3)$$
$$(\mathtt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \Box) \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec 3$$
$$\bullet \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \succ (\mathtt{Fun}\ (g.y.\ x))$$
$$\bullet \triangleright (\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \prec \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x \rangle\!\rangle$$
$$(\mathtt{Ap}\ \Box\ (\mathtt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x \rangle\!\rangle$$
$$(\mathtt{Ap}\ \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x \rangle\!\rangle\ \Box) \triangleright \circ \mid \bullet \succ (\mathtt{N}\ 4)$$
$$(\mathtt{Ap}\ \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x \rangle\!\rangle\ \Box) \triangleright \circ \mid \bullet \prec 4$$
$$\bullet \triangleright \circ \mid y = 4, g = \langle\!\langle g.y.\ x \rangle\!\rangle, x = 3, f = \cdots, \bullet \succ x$$

$$\circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4))$$

$$(\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))$$

$$(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))$$

$$(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle \bullet, f.x.\ (\texttt{Fun}\ (g.y.\ x)) \rangle\!\rangle$$

$$(\texttt{Ap}\ \langle\!\langle \bullet, f \cdots \rangle\!\rangle\ \square) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 3)$$

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$$(\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x \rangle\!\rangle$$

$$(\texttt{Ap}\ \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x \rangle\!\rangle\ \square) \triangleright \circ \mid \bullet \succ (\texttt{N}\ 4)$$

$$(\texttt{Ap}\ \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x \rangle\!\rangle\ \square) \triangleright \circ \mid \bullet \prec 4$$

$$\bullet \triangleright \circ \mid y = 4, g = \langle\!\langle g.y.\ x \rangle\!\rangle, x = 3, f = \cdots, \bullet \succ x$$

$$\bullet \triangleright \circ \mid y = 4, g = \langle\!\langle g.y.\ x \rangle\!\rangle, x = 3, f = \cdots, \bullet \prec 3$$

$\circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))\ (\texttt{N}\ 4))$

$(\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Ap}\ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))\ (\texttt{N}\ 3))$

$(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \succ (\texttt{Fun}\ (f.x.\ (\texttt{Fun}\ (g.y.\ x))))$

$(\texttt{Ap}\ \square\ (\texttt{N}\ 3)) \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid \bullet \prec \langle\!\langle \bullet, f.x.\ (\texttt{Fun}\ (g.y.\ x)) \rangle\!\rangle$

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$\bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \succ (\texttt{Fun}\ (g.y.\ x))$

$\bullet \triangleright (\texttt{Ap}\ \square\ (\texttt{N}\ 4)) \triangleright \circ \mid x = 3, f = \langle\!\langle f \cdots \rangle\!\rangle, \bullet \prec \langle\!\langle (x = 3, f = \cdots, \bullet), g.y.\ x \rangle\!\rangle$

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Refinement

- We already sketched a proof that each C-machine execution has a corresponding M-machine execution (refinement).
- This means any functional correctness (not security or cost) property we prove about all M-machine executions of a program apply just as well to any C-machine executions of the same program.
- Now we want to prove that each E-machine execution has a corresponding C-machine execution (and therefore an M-machine execution).

Ingredients for Refinement

Once again, we want an abstraction function $\mathcal{A}$ that converts E-machine states to C-machine states, such that:

- Each initial state in the E-machine is mapped to an initial state in the C-Machine.
- Each final state in the E-machine is mapped to a final state in the C-Machine.
- For each E-machine transition, either there is a corresponding C-Machine transition, or the two E-machine states map to the same C-machine state.

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- E-Machine values are converted to C-Machine values merely by applying the environment inside closures as a substitution to the expression inside the closure.

With such a function definition, it is trivial to prove that each E-Machine transition has a corresponding transition in the C-Machine, as it is 1:1.

Except!

There is one rule which is not 1:1. Which one?

Semantics Refinement to Program Refinement

The C-Machine and M-machine on these slides have been presented in a very abstract way.

However these program transformations have taken us much closer to an implementation:

- The states (with □ gaps) of the C-Machine are the nodes of the control-flow graph of the program.
- The environments of our E-machine become concrete objects:
  - Stack frames
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And we've learned how to *prove* that our transformations are correct, whether we are adjusting the semantics or the program.

Most compilers do such program-to-program transformations of this kind, but there is no canonical set of them.

An Alternative: A Normalisation

Here is an alternative approach to the C-Machine construction.

First, put the program in A Normal form:

$$2 + (3 * (4 + x))$$

$$\Longrightarrow$$

$$\text{let } v_1 = 4 + x \text{ in let } v_2 = 3 + v_1 \text{ in } 2 + v_2$$

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We can now simplify the use of $\square$ in our C-Machine. How?